# PILLAI COLLEGE OF ENGINEERING, NEW PANVEL
(Autonomous) (Accredited 'A+' by NAAC)
END SEMESTER EXAMINATION
SECOND HALF 2021

BRANCH: FE (EXTC)

Subject: Engineering Mathematics – I
Time: 02.00 Hours
Max. Marks: 60
Date: 04-04-2022

N.B 1. Q.1 is compulsory
2. Attempt any two from the remaining three questions

| Q.1. | Attempt all | M | BT | CO |
|---|---|---|---|---|
| a) | *Show that the matrix* $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is orthogonal and find $A^{-1}$. | 5 | 1,3 | 5 |
| b) | Expand $\sin 5\theta$ in power of $\sin\theta$ and $\cos\theta$. | 5 | 1,3 | 1 |
| c) | Solve for z if $e^{z}=1+i\sqrt{3}$ | 5 | 2 | 1 |
| d) | State Euler's Theorem for two variables, if f(x,y)= $\tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{y}\right]$, find the value of $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$ | 5 | 2 | 4 |
| Q.2. | Attempt all | | | |
| a) | If $u=f(x-y,y-z,z-x)$, find $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$. | 4 | 3,5 | 4 |
| b) | If $6\sinh x+2\cosh x+7=0$, *find* $\tanh x$. | 4 | 3,4 | 2 |
| c) | Find two non-singular matrices P and Q so that PAQ is in normal form where, A = $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ | 6 | 3 | 5 |
| d) | Find an approximate value of the root of the equation $2x-\log x-6=0$ by Bisection Method. | 6 | 4 | 6 |
| Q.3. | Attempt all | | | |
| a) | Solve by Jacobi's method, the equation: 2x+y+4z = 12, 4x+11y-z = 33, 8x-3y+2z = 20 | 4 | 3,5 | 6 |
| b) | Find the $n^{th}$ derivative of $\sinh 2x\,(\sin 2x)^2$ | 4 | 3,4 | 3 |
| c) | Test for consistency of the following equations and solve them if consistent x-2y+3t= 2, 2x+y+z+t= -4, 4x-3y+z+7t=8 | 6 | 3 | 5 |
| d) | Discuss the maxima and minima of $x^3+y^3-3x-12y+40$ | 6 | 4 | 4 |
| Q.4. | Attempt all | | | |
| a) | Expand $\tan x$ in ascending power of x. | 4 | 3,5 | 3 |
| b) | Show that $\vec{F}=(y^2-z^2+3yz-2x)i+(3xz+2xy)j+(3xy-2xz+2z)k$ is irrotational. | 4 P. | 3,4 T. | 4 O. |

| | | | | |
|---|---|---|---|---|
| c) | If x=tan(log $y$) then show that $$(1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0$$ | 6 | 3 | 3 |
| d) | Find the continued product of the roots of $x^4=1+i$ | 6 | 4 | 1 |

CO1-Apply the basic concept of complex numbers and use it to solve problems in engineering.

CO2-Apply the basic concept of Hyperbolic, logarithmic functions in engineering problems.

CO3-Apply the concept of expansion of functions, successive differentiation and vector differentiation in optimization problems.

CO4-Use the basic concepts of partial differentiation in finding the Maxima and Minima required in engineering problems

CO5-Use the concept of matrices in solving the system of equations used in many areas of research...

CO6-Apply the concept of numerical Methods for solving the engineering problems with the help of SCILAB software.

BT Levels: - 1 Remembering ,2 Understanding, 3 Applying,4 Analyzing, 5 Evaluating, 6 Creating.

M-Marks, BT- Bloom's Taxonomy, CO-Course Outcomes.

PCE, New Panvel (Autonomous) QP Code 210051

Sem. Examination $2^{nd}$ Half 2021

Branch - F.Y. B.Tech (Extc)

Sub : E.M. I

D.O.E. 04/04/22

Max Marks - 60

Time 11:00 to 1:00pm

1Q (a) $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

then $A' = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$

$$\therefore AA' = \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA' = I$

$\therefore$ A is orthogonal.

$\therefore A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$

(b) Expand $\sin 5\theta$

We know that $(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$

$$= \cos^5\theta + 5\cos^4\theta\,(i\sin\theta) + 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^5$$

$$\therefore (\cos 5\theta + i\sin 5\theta) = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating imaginary part by both parts, we get

$$\boxed{\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}$$

**1) c** $e^z = 1 + i\sqrt{3}$

Taking log on both sides

$$z = \log(1 + i\sqrt{3})$$

$$\therefore z = \frac{1}{2}\log(1+3) + i\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$z = \frac{1}{2}\log 4 + i\frac{\pi}{3}$$

or, $\boxed{z = \log 2 + i\frac{\pi}{3}}$

**1 d** Euler's Th — If $u$ is a homogeneous function of two variables $x$ & $y$ of degree $n$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Now $f(x,y) = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{y}\right]$ $\quad X = xt,\ Y = yt$

$$\therefore f(X,Y) = \tan^{-1}\left[\frac{t\sqrt{x^2+y^2}}{yt}\right] = f(x,y)$$

$\therefore$ $f(x,y)$ is a H.f. of degree 0.

$$\therefore \boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0}$$

**2 Q** If $u = f(x-y, y-z, z-x)$

let $r = x - y$, $\quad s = y - z$, $\quad t = z - x$

$\therefore u = f(r, s, t)$

$\therefore$ u is a composite function of $x, y$ & $z$.

$$\begin{array}{c} u \\ r \quad s \quad t \\ x, y, z \end{array}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}\cdot(0) + \frac{\partial u}{\partial t}(-1) \quad \text{— (1)}$$

$$\& \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}\cdot(1) + \frac{\partial u}{\partial t}(0) \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$
$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}(1) \quad \text{— } 3$$

$\therefore \boxed{u_x + u_y + u_z = 0}$ [ Eq{1 + 2 + 3} ]

**2b**

$$6\sinh x + 2\cosh x + 7 = 0$$
$$6\left(\frac{e^x - e^{-x}}{2}\right) + 2\left(\frac{e^x + e^{-x}}{2}\right) + 7 = 0$$
$$\frac{6e^x - 6e^{-x} + 2e^x + 2e^x + 14}{2} = 0$$
$$\therefore 8e^x - 4e^{-x} + 14 = 0$$
$$4e^x - 2e^{-x} + 7 = 0$$
$$4e^x - \frac{2}{e^x} + 7 = 0$$
$$\frac{4e^{2x} - 2 + 7e^x}{e^x} = 0$$
$$4e^{2x} + 7e^x - 2 = 0$$
$$4e^{2x} + 8e^x - e^x - 2 = 0$$
$$4e^x(e^x + 2) - (e^x + 2) = 0$$
$$(e^x + 2)(4e^x - 1) = 0$$

$\therefore e^x + 2 = 0$ or, $4e^x - 1 = 0$

$e^x = -2$ $\quad\quad \therefore \boxed{e^x = \frac{1}{4}}$

$\therefore x = \log\frac{1}{4}$

$$\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\frac{1}{4} - 4}{\frac{1}{4} + 4} = \boxed{\frac{-15}{17}}$$

$$\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(-2) - \left(-\frac{1}{2}\right)}{(-2) + \left(-\frac{1}{2}\right)} = \frac{-2 + \frac{1}{2}}{-2 - \frac{1}{2}}$$
$$= \frac{-4+1}{-4-1} = \frac{-3}{-5} = \boxed{\frac{3}{5}}$$

$\underline{\underline{2c}}$ $A_{3\times 4} = I_3 A I_4$

$$\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$
$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 5 & 1 & 4 & -2 \\ 3 & 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 3R_1$
$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 0 & 21 & -51 & 93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + 4C_1$
$C_3 \to C_3 - 11C_1$
$C_4 \to C_4 + 19C_1$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 21 & -51 & 93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -11 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{C_2}{7}$
$C_3 \to \frac{C_3}{-17}$
$\cancel{C_4 \to C_4}$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 93 \\ 0 & 2 & 2 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 4/7 & 11/17 & 19 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{R_2}{3}$
$R_3 \to \frac{R_3}{2}$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 31 \\ 0 & 1 & 1 & 31 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix} A \begin{bmatrix} 1 & 4/7 & 11/17 & 19 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 31 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} A \begin{bmatrix} 1 & 4/7 & 11/7 & 19 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$
$C_4 \to C_4 - 31C_2$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -5/3 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & 1 & 19-31\left(\frac{4}{7}\right) \\ 0 & \frac{1}{7} & -\frac{1}{7} & -31/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$\therefore \rho(A) = 2$$

29 d $f(x) = 2x - \log x - 6$

$\therefore f(0) = -6$

$f(1) = -4$

$f(2) = -2.6931$

$f(3) = -1.0986$

$f(4) = 0.6137$

$\therefore$ Roots lies b/w 3 & 4.

$\therefore a = 3,\ b = 4$

$\therefore$ Ist Iteration $x = \frac{a+b}{2} = \frac{3+4}{2} = \frac{7}{2} = 3.5$

Now $f(3.5) = -0.25276$

Now Roots lies b/w 3.5 & 4.

IInd Iteration $x = \frac{3.5+4}{2} = \frac{7.5}{2} = 3.75$

Now $f(3.75) = 0.17824$

$\therefore$ Root lies b/w 3.5 & 3.75.

IIIrd Iteration $x = \frac{3.5+3.75}{2} = \frac{7.25}{2} = 3.625$

$\therefore f(3.625) = -0.03785$

$\therefore$ Root lies b/w 3.625 & 3.75.

IV Iteration $x = \frac{3.625+3.75}{2} = \frac{7.375}{2} = 3.6875$

$\therefore f(3.6875) = 0.07005$

V Iteration $x = \frac{3.625+3.6875}{2} = \frac{7.3125}{2} = 3.65625$

$\therefore f(3.65625) = 0.016061$

VI Iteration $x = \frac{3.625+3.65625}{2} = \frac{7.28125}{2} = 3.640625$

$\therefore x = 3.64$

3a

$$2x+y+4z=12$$
$$4x+11y-z=33$$
$$8x-3y+2z=20$$

$$x=\frac{20+3y-2z}{8} \quad -(1)$$
$$y=\frac{33-4x+z}{11} \quad -(2)$$
$$z=\frac{12-2x-y}{4} \quad -(3)$$

Ist $x=2.5, \quad y=3, \quad z=3$

IInd $x=\frac{23}{8}=2.875, \quad y=\frac{26}{11}=2.3636, \quad z=1$

IIIrd $x=3.13635, \quad y=\frac{45}{22}=2.045, \quad z=\frac{2429}{2500}=0.9716$

IV $x=3.023975, \quad y=1.9478, \quad z=0.920575$

V $x=3.00028, \quad y=1.9840, \quad z=1.00155$

VI $x=2.9925, \quad y=2, \quad z=1$

$\therefore x=3, \ y=2, \ z=1$

3b

$$y=\sinh 2x \sin^2 2x$$
$$y=\frac{e^{2x}-e^{-2x}}{2}\left(\frac{1-\cos 4x}{2}\right)$$
$$y=\frac{1}{4}\left[e^{2x}(1-\cos 4x)-e^{-2x}(1-\cos 4x)\right]$$
$$y=\frac{1}{4}\left[e^{2x}-e^{2x}\cos 4x-e^{-2x}+e^{-2x}\cos 4x\right]$$
$$\therefore y_n=\frac{1}{4}\Big[2^n e^{2x}-e^{2x}\left\{(20)^{n/2}\cos(4x+n\tan^{-1}(2))\right\}$$
$$-(-2)^n e^{-2x}-e^{-2x}\left\{(20)^{n/2}\cos(4x+n\tan^{-1}(-2))\right\}\Big]$$

**3c** $x - 2y + 0z + 3t = 2$

$2x + y + z + t = -4$

$4x - 3y + z + 7t = 8$

$\therefore$ The matrix

$$[A : B] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 & -4 \\ 4 & -3 & 1 & 7 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 5 & 1 & -5 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$\therefore \rho(A : B) = 3$

But $\rho(A) = 2$

$\therefore \rho(A : B) \neq \rho(A)$

$\therefore$ Inconsistent.

**3d** $f(x,y) = y = x^3 + y^3 - 3x - 12y + 40$

$\therefore \dfrac{\partial y}{\partial x} = \dfrac{\partial f}{\partial x} = 3x^2 - 3$ & $\dfrac{\partial f}{\partial y} = 3y^2 - 12$

for max & min value of $f$,

$\dfrac{\partial f}{\partial x} = 0$ , $\dfrac{\partial f}{\partial y} = 0$

$3x^2 - 3 = 0$ , $3y^2 - 12 = 0$

$3x^2 = 3$ , $y^2 = 4$

$x^2 = 1$ , $y = \pm 2$

$\therefore x = \pm 1$

$\therefore$ St. pts are $(1, 2)$, $(1, -2)$, $(-1, 2)$, $(-1, -2)$

| St. pts | $r = \frac{\partial^2 f}{\partial x^2} = 6x$ | $s = \frac{\partial^2 f}{\partial x \partial y} = 0$ | $t = \frac{\partial^2 f}{\partial y^2} = 6y$ | $rt - s^2 = 36xy$ | Remark |
|---|---|---|---|---|---|
| $(1, 2)$ | 6 | 0 | 12 | $72 > 0$ | minima |
| $(1, -2)$ | 6 | 0 | $-12$ | $-72 < 0$ | neither max nor min |
| $(-1, 2)$ | $-6$ | 0 | 12 | $-72 < 0$ | neither maxima nor min |
| $(-1, -2)$ | $-6$ | 0 | $-12$ | $72 > 0$ | maxima |

$\therefore$ $f_{max}$ at $(-1,-2)$ is 58.
& $f_{min}$ at $(1,2)$ is 22.

**4a** Expansion of $\tan x$

Let $y = \tan x$ $\qquad \therefore y(0) = 0$

$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \qquad y_1(0) = 1$

$y_2 = 2y\,y_1 \qquad y_2(0) = 0$

$y_3 = 2y_1^2 + 2y\,y_2 \qquad y_3(0) = 2$

$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3$

$\qquad \& \; y_4(0) = 0$

$y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4$
$\quad = 6y_2^2 + 8y_1y_3 + 2yy_4 \qquad y_5(0) = 15$

$\because$ Maclaurin's Series is

$$y = y(0) + x\,y_1(0) + \frac{x^2}{\lfloor 2}y_2(0) + \frac{x^3}{\lfloor 3}y_3(0) + \frac{x^4}{\lfloor 4}y_4(0) + \frac{x^5}{\lfloor 5}y_5(0) + \cdots$$

$$\therefore \tan x = 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(2) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(15) + \cdots$$

$$\therefore \boxed{\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots}$$

**4b** S.T. $\vec{F}$ is irrotational {if curl $\vec{F} = 0$}

$$\therefore \text{curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\,[3x - 3x] - \hat{j}\,[3y - 2z + 2z - 3y] + \hat{k}\,[3z + 2y - 2y - 3$$

$$= 0$$

$\therefore$ $\vec{F}$ is irrotational.

4c. If $x = \tan(\log y)$

then $\tan^{-1} x = \log y$

$\therefore \quad y = e^{\tan^{-1} x}$

$y_1 = \dfrac{e^{\tan^{-1} x}}{1+x^2}$

$\therefore \quad y_1 = \dfrac{y}{1+x^2}$

or, $(1+x^2) y_1 = y$

Again diff. w.r.t $x$

$(1+x^2) y_2 + 2x y_1 = y_1$

Now differentiating again w.r.t. $x$, $n$ times, we get

$$\left[(1+x^2) y_{n+2} + n y_{n+1}(2x) + \frac{n(n-1)}{|2} y_n (2)\right] + 2\left[x y_{n+1} + n y_n (1)\right] = y_{n+1}$$

$$\therefore \boxed{(1+x^2) y_{n+2} + [(2n+1)x - 1] y_{n+1} + n(n+1) y_n = 0}$$

4d

$x^4 = 1 + i$

$\therefore \quad x = (1+i)^{1/4}$

$x = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)\right]^{1/4}$

$x = \left[(\sqrt{2})^{1/4}\left\{\cos\left(\dfrac{2k\pi + \frac{\pi}{4}}{4}\right) + i \sin\left(\dfrac{2k\pi + \frac{\pi}{4}}{4}\right)\right\}\right]$

$\therefore \quad x = 2^{1/8}\left[\cos\left(\dfrac{8k+1}{16}\right)\pi + i \sin\left(\dfrac{8k+1}{16}\right)\pi\right]$

where $k = 0, 1, 2, 3$

$\therefore \quad x_0 = 2^{1/8}\left[\cos\frac{\pi}{16} + i \sin\frac{\pi}{16}\right]$, $x_1 = 2^{1/8}\left[\cos\frac{9\pi}{16} + i \sin\frac{9\pi}{16}\right]$

$x_2 = 2^{1/8}\left[\cos\frac{17\pi}{16} + i \sin\frac{17\pi}{16}\right]$, $x_3 = 2^{1/8}\left[\cos\frac{25\pi}{16} + i \sin\frac{25\pi}{16}\right]$

$\therefore$ Continued product $= x_0 x_1 x_2 x_3$

$= 2^{4/8}\left[\cos\left\{\frac{\pi}{16} + \frac{9\pi}{16} + \frac{17\pi}{16} + \frac{25\pi}{16}\right\} + i \sin\left\{\frac{\pi}{16} + \frac{9\pi}{16} + \frac{17\pi}{16} + \frac{25\pi}{16}\right\}\right]$

$= 2^{1/2}\left[\cos\frac{52\pi}{16} + i \sin\frac{52\pi}{16}\right] = \sqrt{2}\left[\cos\frac{13\pi}{4} + i \sin\frac{13\pi}{4}\right]$

$= \sqrt{2}\left[\cos\left(3\pi + \frac{\pi}{4}\right) + i \sin\left(3\pi + \frac{\pi}{4}\right)\right] = \sqrt{2}\left[-\cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\right]$

$= \sqrt{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$