Branch - EXTC Q.P. Code 210084

Sub :- GM - I

Q1 a) $x^7 + x^4 + i(x^3+1) = 0$

$x^4(x^3+1) + i(x^3+1) = 0$

$(x^3+1)(x^4+i) = 0$

$x^3+1 = 0$ or $x^4+i = 0$

$x^3 = -1$

$x^3 = \cos\pi + i\sin\pi$

$x = [\cos(2K+1)\pi + i\sin(2K+1)\pi]^{1/3}$

$x = \cos(2K+1)\frac{\pi}{3} + i\sin(2K+1)\frac{\pi}{3}$

when $K = 0, 1, 2.$

For, $x^4 + i = 0$

$x^4 = -i$

$x^4 = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$x = [\cos(2K+\frac{1}{2})\pi - i\sin(2K+\frac{1}{2})\pi]^{1/4}$

$x = \cos(4K+1)\frac{\pi}{8} - i\sin(4K+1)\frac{\pi}{8}$

when $K = 0, 1, 2, 3.$

1 (b) Let $\tanh^{-1}x = y$, $\therefore$ $x = \tanh y$

Now $\frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\operatorname{sech} y}$

$= \sinh y$

$\therefore$ $y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$\Rightarrow$ $\tanh^{-1}x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

1 (c) Given $f(x) = e^x$, $f(0) = 1$

$f'(x) = e^x$, $f'(0) = 1$

$f''(x) = e^x$, $f''(0) = 1$

$f'''(x) = e^x$, $f'''(0) = 1$

$f^{iv}(x) = e^x$, $f^{iv}(0) = 1$

By Maclaurin's Series

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

1(d) Given, $u = f(r)$, $r = \sqrt{x^2+y^2}$

$$\therefore r^2 = x^2+y^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now, $\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r} + f'(r)\left[\frac{r(1) - x\,\partial r/\partial x}{r^2}\right]$$

$$= \frac{x^2}{r^2}f''(r) + f'(r)\left[\frac{r - x^2/r}{r^2}\right]$$

$$= \frac{x^2}{r^2}f''(r) + f'(r)\left[\frac{r^2-x^2}{r^3}\right]$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}f''(r) + f'(r)\left[\frac{r^2-y^2}{r^3}\right]$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{x^2+y^2}{r^2}\right)f''(r) + f'(r)\left[\frac{r^2-x^2-y^2+r^2}{r^3}\right]$$

$$= \frac{r^2}{r^2}f''(r) + f'(r)\left[\frac{2r^2-r^2}{r^3}\right]$$

$$= f''(r) + \frac{1}{r}f'(r)$$

Q.2 a) Let $x+iy = (1+i\sqrt{3})^{1+i\sqrt{3}}$

Taking log on both side, we get.

$$\log(x+iy) = (1+i\sqrt{3})\log(1+i\sqrt{3})$$

$$= (1+i\sqrt{3})[\log 2 + i\pi/3]$$

$$= \left(\log 2 - \frac{\pi}{\sqrt{3}}\right) + i\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$$

$$x+iy = e^{\log 2} e^{-\pi/\sqrt{3}} \cdot e^{i(\pi/3 + \sqrt{3}\log 2)}$$

$$= 2e^{-\pi/\sqrt{3}}\left[\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right]$$

$\therefore$ Real Part is

$$x = 2e^{-\pi/\sqrt{3}}\left(\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right)$$

2.b) $\bar{F} = ye^{xy}\cos z\, i + xe^{xy}\cos z\, j - e^{xy}\sin z\, k.$

$$\because \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{xy}\cos z & xe^{xy}\cos z & -e^{xy}\sin z \end{vmatrix}$$

$$= i\left[-xe^{xy}\sin z + xe^{xy}\sin z\right] - j\left[-ye^{xy}\sin z + ye^{xy}\sin z\right]$$

$$+ k\left[e^{xy}\cos z + xye^{xy}\cos z - e^{xy}\cos z - xye^{xy}\cos z\right]$$

$$= i(0) + j(0) + k(0) = 0$$

$\therefore$ $\bar{F}$ is irrotational.

2(c) $2\cos\theta = x + \frac{1}{x}$, $2i\sin\theta = x - \frac{1}{x}$

Now,

$$(2i\sin\theta)^4 (2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x - \frac{1}{x}\right)\left(x^6 - 3x^2 + 3\frac{1}{x^2} - \frac{1}{x^6}\right)$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= x^7 + \frac{1}{x^7} - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right) - \left(x^5 + \frac{1}{x^5}\right)$$

$$2^7 \sin^4\theta \cos^3\theta = 2\cos 7\theta - 3(2\cos 3\theta) + 3(2\cos\theta) - (2\cos 5\theta)$$

$$\sin^4\theta \cos^3\theta = \frac{1}{2^6}(\cos 7\theta - 3\cos 3\theta + 3\cos\theta - \cos 5\theta)$$

$$= \frac{1}{64}(\cos 7\theta - 3\cos 3\theta + 3\cos\theta - \cos 5\theta)$$

2(d) $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$

$$A^\theta = (\bar{A})' = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

Now, $AA^\theta = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix} \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4-2i \end{bmatrix}$

$$= \begin{bmatrix} 4-i^2+9+(-1)^2-9i^2 & -10-5i-3i+14i+2 \\ -\cancel{10}+5i+3i-\cancel{10}-10i & 25+1+16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -8+6i \\ -2i & 46 \end{bmatrix} \neq I$$

$\therefore$ A is not unitary.

Q. 3 (a) $\quad y = \tan^{-1} x$

$$y_1 = \frac{1}{1+x^2}$$

$$(1+x^2) y_1 = 1 \quad \text{------} (1)$$

$$\cancel{(1+x^2) y_2 + 2x y_1 = 0}$$

Applying Leibnitz theo. on eq^n (1), we get

$$(1+x^2) y_{n+1} + n(2x) y_n + \frac{n(n-1)}{2!}(2) y_{n-1} = 0$$

$$(x^2+1) y_{n+1} + 2xn y_n + n(n-1) y_{n-1} = 0$$

3 (b) $\quad U = \sinh^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$

$\Rightarrow \sinh U = \frac{x^3+y^3}{x^2+y^2}$

Putting $X = xt$ & $Y = yt$ we get

$\sinh U = t\left(\frac{x^3+y^3}{x^2+y^2}\right)$

$\Rightarrow$ U is not homogeneous, but $f(U) = \sinh U$ is homo. of degree 1.

$\therefore$ By corollary of Euler's theo,

$$x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x \partial y} + y^2\frac{\partial^2 U}{\partial y^2} = g(U)[g'(U)-1]$$

when $g(U) = n\frac{f(U)}{f'(U)} = 1\,\frac{\sinh U}{\cosh U} = \tanh U$

$\therefore g'(U) = \text{sech}^2 U$

$$\Rightarrow x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x \partial y} + y^2\frac{\partial^2 U}{\partial y^2} = \tanh U[\text{sech}^2 U - 1]$$

$$= -\tanh^3 U$$

3 (c) $x+y+z=1$

$x+2y+4z=\lambda$

$x+4y+10z=\lambda^2$

The matrix eq^n is

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda-1 \\ \lambda^2-1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda-1 \\ \lambda^2-3\lambda+2 \end{bmatrix}$$

The given eq^n will be consistent if the rank(A) = rank$[A:B]$

This requires $\lambda^2-3\lambda+2=0 \Rightarrow \lambda = 2$ or 1.

3 (d) Given, $f(x) = x^3 - 9x^2 - 18$

Now, $f(9) = -18$, $f(10) = 82$

$\Rightarrow$ Root lies between 9 & 10

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_0 = 9$ & $x_{n+1} = x_n - \dfrac{x_n^3 - 9x_n^2 - 18}{3x_n^2 - 18x_n}$

$$x_1 = 9 - \frac{9^3 - 9(9)^2 - 18}{3(9)^2 - 18(9)} = 9.2222$$

$$x_2 = 9.2222 - \frac{(9.2222)^3 - 9(9.2222)^2 - 18}{3(9.2222)^2 - 18(9.2222)} = 9.2121$$

$$x_3 = 9.2121 - \frac{(9.2121)^3 - 9(9.2121)^2 - 18}{3(9.2121)^2 - 18(9.2121)} = 9.2121$$

$\because$ x_2 & x_3 gives same value.

$\therefore$ Root of an eq^n is 9.2121

Q 4 (a) $u + iv = (1+\cos\theta + i\sin\theta)(1+\cos 2\theta + i\sin 2\theta)$

$= (2\cos^2\theta/_2 + i\, 2\sin\theta/_2 \cos\theta/_2)(2\cos^2\theta + 2i\sin\theta\cos\theta)$

$= 2\cos\theta/_2 (\cos\theta/_2 + i\sin\theta/_2) \cdot 2\cos\theta (\cos\theta + i\sin\theta)$

$= 4\cos\theta/_2 \cdot \cos\theta (e^{i\theta/2} \cdot e^{i\theta})$

$= 4\cos\theta\cos\theta/_2 \; e^{i3\theta/2}$

$u + iv = 4\cos\theta\cos\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$

equating Real & Imaginary Part we get

$U = 4\cos\theta/_2 \cos\theta \cos 3\theta/_2$

$V = 4\cos\theta/_2 \cos\theta \sin 3\theta/_2$

$$\therefore U^2 + V^2 = 16\cos^2\frac{\theta}{2}\cos^2\theta\cos^2\frac{3\theta}{2} + 16\cos^2\frac{\theta}{2}\cos^2\theta\sin^2\frac{3\theta}{2}$$

$$= 16\cos^2\theta\cos^2\frac{\theta}{2}\left(\cos^2\frac{3\theta}{2} + \sin^2\frac{3\theta}{2}\right)$$

$$= 16\cos^2\theta\cos^2\frac{\theta}{2}$$

4(b) $\log\tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = \log\left[\frac{1 + \tan(ix/_2)}{1 - \tan(ix/_2)}\right]$

$$= \log\left[\frac{1 + i\tanh x/_2}{1 - i\tanh x/_2}\right]$$

$$= \log\left[1 + i\tanh x/_2\right] - \log\left[1 - i\tanh x/_2\right]$$

$$= \frac{1}{2}\log\left(1 + \tanh^2 x/_2\right) + i\tan^{-1}\tanh\left(\frac{x}{2}\right)$$

$$- \frac{1}{2}\log\left(1 + \tanh^2\frac{x}{2}\right) + i\tan^{-1}\tanh\left(\frac{x}{2}\right)$$

$$= 2i\tan^{-1}\left(\tanh x/_2\right) \quad \left[\text{Putting } \alpha = \tanh x/_2\right]$$

$$= i\tan^{-1}\left(\frac{2\tanh x/2}{1 - \tanh^2 x/2}\right)$$

$$= i\tan^{-1}\sinh x$$

4(c) $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$f_x = 3x^2 + 3y^2 - 6x = 0$ —— (1)

$f_y = 6xy - 6y = 0$ —— (2)

$\Rightarrow 6y(x-1) = 0 \Rightarrow y = 0$ or $x = 1$

from (1) ~~Now~~, when $y = 0$, we get

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0,\ x = 2$$

$\Rightarrow (0,0)\ (2,0)$

from (1), when $x = 1$, we get

$$3 + 3y^2 - 6 = 0$$
$$3y^2 - 3 = 0$$
$$y^2 = 1 \Rightarrow y = \pm 1$$

$\Rightarrow (1,1)\ \&\ (1,-1)$

$\therefore (0,0)\ (2,0)\ (1,1)\ \&\ (1,-1)$ are the stationary Point.

Now, $r = f_{xx} = 6x - 6$

$s = f_{xy} = 6y$

$t = f_{yy} = 6x - 6$

~~$rt - s^2 =$~~

Pt.	r	s	t	$rt-s^2$	Result
$(0,0)$	-6	0	-6	$36 > 0$	Max.
$(2,0)$	6	0	6	$36 > 0$	Mini
$(1,1)$	0	6	0	$-36 < 0$	Neither max or min.
$(1,-1)$	0	-6	0	$-36 < 0$	Neither max or min.

$\therefore$ Max value $= 4$

Min value $= (2)^3 - 3(2)^2 + 4 = 0$

4(d) $5x - y - z = 10$

$2x + 4y = 12$

$x + y + 5z = -1$

$$\Rightarrow \quad x = \frac{10 + y + z}{5}$$

$$y = \frac{12 - 2x}{4}$$

$$z = \frac{-1 - x - y}{5}$$

Consider $x_0 = 2$, $y_0 = 2$, $z_0 = 1$.

I iteraⁿ.

$$x_1 = \frac{10 + (2) + (1)}{5} = 2.6$$

$$y_1 = \frac{12 - 2(2.6)}{4} = 1.7$$

$$z_1 = \frac{-1 - 2.6 - 1.7}{5} = -1.06$$

II iteraⁿ.

$$x_2 = \frac{10 + (1.7) + (-1.06)}{5} = 2.552$$

$$y_2 = \frac{12 - 2(2.552)}{4} = 1.724$$

$$z_2 = \frac{-1 - (2.552) - (1.724)}{5} = -1.0552$$

III iteraⁿ

$$x_3 = \frac{10 + 1.724 + (-1.0552)}{5} = 2.1338$$

$$y_3 = \frac{12 - 2(2.1338)}{4} = 1.9331$$

$$z_3 = \frac{-1 - 2.1338 - 1.9331}{5} = -1.0134$$

IV iteraⁿ $x_4 = \boxed{2.1839}$, $y_4 = \boxed{1.9081}$, $z_4 = \boxed{-1.0184}$