

Solution of (EM IV) EXTC.  
Supplementary Examination.

Q1)

a) Let  $u$  &  $v$  be two vectors then Cauchy Schwartz inequality is given by  $|u \cdot v| \leq \|u\| \cdot \|v\|$ .

$$\text{Let } u = (a, b, c) \text{ \& } v = (1, 1, 1)$$

$$\therefore u \cdot v = a + b + c.$$

$$|u \cdot v| = a + b + c.$$

$$\|u\| = \sqrt{a^2 + b^2 + c^2} \quad \|v\| = \sqrt{3}$$

$\therefore$  By Cauchy Schwartz inequality

$$|u \cdot v| \leq \|u\| \cdot \|v\|$$

$$a + b + c \leq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{3}$$

$$\therefore (a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$$

(b) Given:-  $T_0 = 0$  is transmitted  $T_1 = 1$  is transmitted

$$P(T_0) = 0.4$$

$$\therefore P(T_1) = 1 - P(T_0) = 1 - 0.4 = 0.6$$

Let  $R_0 = 0$  is received &  $R_1 = 1$  is received.

$R_1 = 1$  is received &  $R_0 = 0$  is received.

$$\therefore P(R_0 | T_0) = 0 \text{ is received when } 0 \text{ is transmitted} = 0.95$$

$$P(R_1 | T_0) = 1 \text{ is received when } 0 \text{ is transmitted} = 1 - 0.95 = 0.05$$

$$P(R_1 | T_1) = 1 \text{ is received when } 1 \text{ is transmitted} = 0.90$$

$$P(R_0 | T_1) = 0 \text{ is received when } 1 \text{ is transmitted} = 1 - 0.90 = 0.1$$



$$(iv) F = xy + y^2 - 2y^2y'$$

F contain both  $x, y, y'$ .

∴ By Euler's equation we have.

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0$$

$$[x + 2y - 4yy'] - \frac{d}{dx} (2y^2) = 0$$

$$x + 2y - 4yy' + 4yy' = 0$$

$$x + 2y = 0$$

$$y = -x/2$$

Q2)

a)  $\lambda = 20$ 

$$P = 0.05$$

$$Q = 1 - P = 1 - 0.05 = 0.95$$

$$M = nP$$

$$= 20 \times 0.05 = 1$$

$$n = 20$$

$\therefore X_i = \text{No. of Defective}$

$P(\text{Packets containing exactly 2})$  and  $P(\text{at least 2 defective}) = ?$

$\Rightarrow$

$$i) P(\text{Packets containing exactly 2}) = P(X=2)$$

$$= {}^n C_x p^x q^{n-x}$$

$$= {}^{20} C_2 (0.05)^2 (0.95)^{18}$$

$$= {}^{20} C_2 (0.05)^2 (0.95)^{18}$$

$$= 0.19$$

$$ii) P(\text{at least 2 defective}) = 1 - P(X \leq 1)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [{}^n C_0 p^0 q^{n-0} + {}^n C_1 p^1 q^{n-1}]$$

$$= 1 - [{}^{20} C_0 (0.05)^0 (0.95)^{20} + {}^{20} C_1 (0.05)^1 (0.95)^{19}]$$

$$= 1 - [0.36 + 0.38]$$

$$= 0.26$$

$\therefore$  No. of packets containing exactly 2 defective =  $N \cdot P$

$$= 1000 \times 0.19$$

$$= 190$$

No. of packets containing at least 2 defective =  $N \cdot P$

$$= 1000 \times 0.26$$

$$= 260$$

By poisson distribution

$$P(X=2) = \frac{e^{-1} 1^2}{2!} = 0.1839$$

$$\text{No. of exactly 2 defective} = 183$$

$$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left( \frac{e^{-1} 1^0}{0!} + \frac{e^{-1} 1^1}{1!} \right) =$$

$$Q2b) f(z) = \frac{2z-3}{z^2-4z-3}$$

$$= \frac{2z-3}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$\therefore \Rightarrow 2z-3 = A(z-3) + B(z-1)$$

$$\text{put } z=1$$

$$A = \frac{1}{2}$$

$$\text{put } z=3$$

$$B = \frac{3}{2}$$

$$\therefore f(z) = \frac{1/2}{z-1} + \frac{3/2}{z-3}$$

$$= \frac{1}{2} \left[ \frac{1}{z-1} + \frac{3}{z-3} \right] = \frac{1}{2} \left[ \frac{1}{[(z-4)+3]} + \frac{3}{[(z-4)+1]} \right]$$

case i) when  $|z-4| < 1$ .

$$f(z) = \frac{1}{2} \left[ \frac{1}{3 \left[ 1 + \frac{z-4}{3} \right]} + \frac{3}{[1 + (z-4)]} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{3} [1 + (\frac{z-4}{3})]^{-1} + 3 [1 + (z-4)]^{-1} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{3} \left( 1 - \left(\frac{z-4}{3}\right) + \left(\frac{z-4}{3}\right)^2 + \dots \right) + 3 \left( 1 - (z-4) + (z-4)^2 - (z-4)^3 + \dots \right) \right]$$

$$= \frac{1}{6} \left[ 1 - \left(\frac{z-4}{3}\right) + \left(\frac{z-4}{3}\right)^2 + \dots \right] + \frac{3}{2} \left[ 1 - (z-4) + (z-4)^2 + \dots \right]$$

Reversed Taylor series.

case ii)  $|z-4| > 3$

$$f(z) = \frac{1}{2} \left[ \frac{1}{[(z-4)+3]} + \frac{3}{[(z-4)+1]} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{(z-4) \left[ 1 + \frac{3}{z-4} \right]} + \frac{3}{(z+4) \left[ 1 + \frac{1}{z-4} \right]} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{z-4} \left[ 1 + \frac{3}{z-4} \right]^{-1} + \frac{3}{z-4} \left[ 1 + \frac{1}{z-4} \right]^{-1} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{z-4} \left( 1 + \frac{3}{z-4} + \left(\frac{3}{z-4}\right)^2 + \left(\frac{3}{z-4}\right)^3 + \dots \right) + \frac{3}{z-4} \left[ 1 - \frac{1}{z-4} + \left(\frac{1}{z-4}\right)^2 - \left(\frac{1}{z-4}\right)^3 + \dots \right] \right]$$

This is Required Laurent series.

$$(c) \int_0^1 (y'^2 - 4y^2 + 2xy) dx.$$

$$\text{Let } y = c_0 + c_1 x + c_2 x^2 \quad y(0) = 0 \Rightarrow c_0 = 0.$$

$$y' = c_1 + 2c_2 x \quad y(1) = 0 \Rightarrow c_0 + c_1 + c_2 = 0.$$

$$\therefore y' = c_1 - 2c_2 x.$$

$$y = c_1(x - x^2)$$

$$= c_1(1 - 2x).$$

$$\therefore I = \int_0^1 [c_1^2(1-2x)^2 + 4(x-x^2)^2 c_1^2 + 2x^2 c_1(x-x^2)] dx$$

$$= \int_0^1 c_1^2 (1 - 4x + 4x^2) + 4(x^2 - 2x^3 + x^4) c_1^2 + 2c_1(x^3 - x^4) dx.$$

$$= c_1 \left[ c_1 \left( x - \frac{4x^2}{2} + \frac{4x^3}{3} \right) + 4c_1 \left( \frac{x^3}{3} + \frac{2x^4}{4} + \frac{x^5}{5} \right) + 2 \left( \frac{x^4}{4} - \frac{x^5}{5} \right) \right]_0^1$$

$$= c_1 \left[ c_1 \left( 1 - \frac{4}{2} + \frac{4}{3} \right) + 4c_1 \left( \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + 2 \left( \frac{1}{4} - \frac{1}{5} \right) - 0 \right]$$

$$= c_1 \left[ \frac{c_1}{3} + \left( \frac{-8c_1}{15} \right) + \frac{1}{10} \right]$$

$$I \Rightarrow -\frac{c_1^2}{5} + \frac{1}{10} c_1$$

$$\frac{dI}{dc_1} = 0 \Rightarrow -\frac{2c_1}{5} + \frac{1}{10} = 0 \Rightarrow \frac{2c_1}{5} = \frac{1}{10}$$

$$c_1 = \frac{5}{20} = \frac{1}{4}$$

$$\therefore y = \frac{1}{4}[1-2x] //$$

Q3)

Marks X	Stats Y	$x = x - \bar{x}$ $x = x - 65$	$y = y - \bar{y}$ $y = y - 66$	xy	$x^2$	$y^2$
78	84	13	18	234	169	324
36	51	-29	-15	435	841	225
98	91	33	25	825	1089	625
25	60	-40	-6	240	1600	36
75	68	10	2	20	100	64
82	62	17	-4	-68	289	16
90	86	25	20	500	625	400
62	58	-3	-8	24	9	64
65	53	0	-13	0	0	169
39	47	-26	-19	494	676	361
		$\frac{0}{0}$	$\frac{0}{0}$	$\frac{2704}{5398}$	$\frac{2224}{2224}$	

N=10

$$\bar{x} = \frac{\sum x}{10} = 65$$

$$\bar{y} = \frac{660}{10} = 66$$

$$\therefore r = \frac{N \sum xy - \sum x \sum y}{\sqrt{N \sum x^2 - (\sum x)^2} \cdot \sqrt{N \sum y^2 - (\sum y)^2}}$$

$$= \frac{10(2704) - 0}{\sqrt{10(5398)} \cdot \sqrt{10(2224)}}$$

$$= 0.7804$$

∴ For Spearman's Rank Correlation.

X	Y	R <sub>1</sub>	R <sub>2</sub>	d <sub>i</sub> <sup>2</sup> = (R <sub>1</sub> - R <sub>2</sub> ) <sup>2</sup>
78	84	4	3	1
36	57	9	9	0
98	91	1	1	0
25	60	10	6	1.6
75	68	5	4	1
82	62	3	5	4
90	86	2	2	0
62	58	7	7	0
65	53	6	8	4
39	47	8	10	4
				30.

∴  $\sum d_i^2 = 30$ .

$$R = 1 - \frac{6 \sum d_i^2}{N^3 - N}$$

$$= 1 - \frac{6(30)}{10^3 - 10} = 0.8181$$



$$(b) \quad u_1 = (1, 2, 1) \quad u_2 = (1, 1, 3) \quad u_3 = (2, 1, 1)$$

$$\text{Let } v_1 = u_1.$$

$$\therefore v_1 = (1, 2, 1) \quad \|v_1\| = \sqrt{6}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$$

$$= (1, 1, 3) - \frac{\langle (1, 1, 3) \cdot (1, 2, 1) \rangle}{6} (1, 2, 1)$$

$$= (1, 1, 3) - \frac{6}{6} (1, 2, 1)$$

$$= (1, 1, 3) - (1, 2, 1)$$

$$v_2 = (0, -1, 2)$$

$$\|v_2\| = \sqrt{5}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} \cdot v_2$$

$$= (2, 1, 1) - \frac{\langle (2, 1, 1), (1, 2, 1) \rangle}{6} v_1 - \frac{\langle (2, 1, 1), (0, -1, 2) \rangle}{5} v_2$$

$$= (2, 1, 1) - \frac{5}{6} (1, 2, 1) - \frac{1}{5} (0, -1, 2)$$

$$= (2, 1, 1) - \left(\frac{5}{6}, \frac{10}{6}, \frac{5}{6}\right) - \left(0, -\frac{1}{5}, \frac{2}{5}\right)$$

$$= \left(\frac{7}{6}, -\frac{7}{15}, \frac{7}{30}\right)$$

$$\|v_3\| = \sqrt{\frac{49}{30}} = \frac{7}{\sqrt{30}}$$

$\therefore$  orthonormal vectors are.

$$e_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$e_2 = \frac{v_2}{\|v_2\|} = \left( 0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$e_3 = \frac{v_3}{\|v_3\|} = \left( \frac{\sqrt{30}}{6}, -\frac{\sqrt{30}}{15}, \frac{\sqrt{30}}{30} \right) = \left( \frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right)$$

(3)(e) By divergence theorem.

$$\iiint_S \vec{H} \cdot \vec{F} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$$

Here  $F = x^3 i + x^2 y j + x^2 z k$

$$\nabla \cdot F = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)$$

$$= 3x^2 + x^2 + x^2$$

$$= 5x^2$$

$$\therefore \iiint_V \nabla \cdot F \, dV = \iiint_V 5x^2 \, dx \, dy \, dz$$

To cover whole cylinder bounded  $x^2 + y^2 = a^2$ ,  $z=0$  to  $z=b$   
Varies from 0 to b, y varies from  $-\sqrt{a^2 - x^2}$  to  $\sqrt{a^2 - x^2}$   
and x varies from  $-a$  to  $a$ .

$$\therefore \iiint_V 5x^2 \, dx \, dy \, dz = \int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_0^b 5x^2 \, dz \, dy \, dx$$

$$= 5 \int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} x^2 (b) \, dy \, dx$$

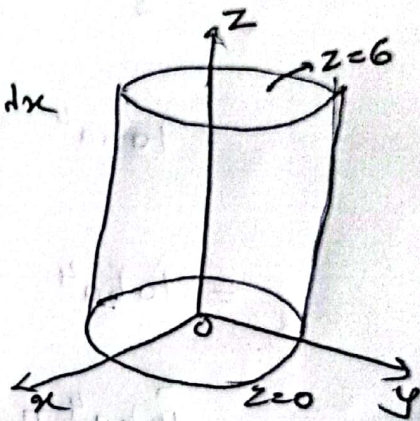
$$= 5b \int_{-a}^a x^2 (y) \frac{\sqrt{a^2 - x^2}}{-\sqrt{a^2 - x^2}} \, dx$$

$$= 5b \int_{-a}^a x^2 \cdot 2 \sqrt{a^2 - x^2} \, dx$$

$$= 10b \int_{-a}^a x^2 \sqrt{a^2 - x^2} \, dx$$

Put  $x = at$   $dx = a \, dt$

when  $x = -a \Rightarrow t = -1$   
 $x = a \Rightarrow t = 1$



$$\iiint_V 5x^2 dV = 10b \int_{-1}^1 a^4 t^2 \sqrt{a^2 - a^2 t^2} a dt$$

$$= 20b \int_0^1 a^4 t^2 \sqrt{1-t^2} dt$$

$$= 20ba^4 \int_0^1 t^2 \sqrt{1-t^2} dt$$

Put  $t^2 = u$

$2t dt = du \Rightarrow dt = \frac{du}{2\sqrt{u}}$   
 limit remain same.

$$\iiint_V 5x^2 dV = \int_0^1 20ba^4 \int_0^1 u \sqrt{1-u} \frac{du}{2\sqrt{u}}$$

$$= \frac{20ba^4}{2} \int_0^1 u^{1/2} (1-u)^{1/2} du = 10ba^4 \beta(3/2, 3/2)$$

$$= 10ba^4 \frac{\Gamma(3/2) \Gamma(3/2)}{\Gamma(3)}$$

$$\therefore \int_0^1 x^m (1-x)^n dx = \frac{\Gamma(m+1) \Gamma(n+1)}{\Gamma(m+n+2)}$$

$$= 10ba^4 \frac{1/2 \cdot 1/2 \cdot \pi}{2}$$

$$= \frac{5ba^4 \pi}{8}$$

$$\iint_S \mathbf{N} \cdot \mathbf{F} dS = \frac{5a^4 b \pi}{4}$$

Q4)

(a)  $\int F \cdot dr = ?$

$F = \cos y i - x \sin y j$

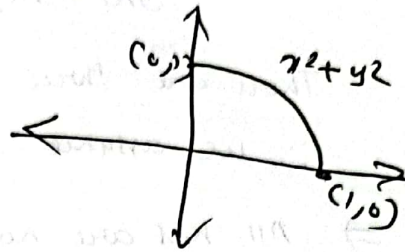
$C$  is the curve  $y = \sqrt{1-x^2}$   
 $x^2 + y^2 = 1$

$\therefore \int F \cdot dr = \int_C (\cos y i - x \sin y j) (dx i + dy j)$

$= \int_C \cos y dx - x \sin y dy$

$= [x \cos y - 0]_{(1,0)}^{(0,1)}$

$= (0 - 1) = -1$



(b) i)  $\{1, x, 1+x+x^2\}$

Soln Let  $v_1 = 1, v_2 = x, v_3 = 1+x+x^2$

Consider

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$k_1 + k_2 x + k_3 (1+x+x^2) = 0 + 0 \cdot x + 0 \cdot x^2$

$k_1 + k_2 x + k_3 + k_3 x + k_3 x^2 = 0 + 0 \cdot x + 0 \cdot x^2$

$(k_1 + k_3) + (k_2 + k_3)x + k_3 x^2 = 0 + 0 \cdot x + 0 \cdot x^2$

$\Rightarrow k_3 = 0$

$\Rightarrow k_2 + k_3 = 0$

$\Rightarrow k_2 = 0$

$\Rightarrow k_1 + k_3 = 0$

$\Rightarrow k_1 = 0$

All  $k_i$ 's are zero.

Hence  $\{1, x, 1+x+x^2\}$  is linearly independent

ii)  $v_1 = (2, 3) + v_2 = (-4, 3) \quad v_3 = (1, 5)$

consider  $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$k_1(2, 3) + k_2(-4, 3) + k_3(1, 5) = (0, 0)$$

$$2k_1 - 4k_2 + k_3 = 0$$

$$3k_1 + 3k_2 + 5k_3 = 0$$

There are three variable + two equation

$\therefore$  we assume  $k_3 = t, t \neq 0$ .

$\Rightarrow$  All  $k_i$ 's are not zero.

Hence  $\{v_1, v_2, v_3\}$  are linearly dependent.

(c)  $\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2\cos\theta + a^2} d\theta, \quad -1 < a < 1.$

soln let  $z = e^{i\theta} \quad \therefore dz = ie^{i\theta} d\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \frac{z + \bar{z}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

consider  $\int_0^{2\pi} \frac{e^{2i\theta}}{1 - 2\cos\theta + a^2} d\theta = \int \frac{z^2}{1 - 2a\left(\frac{z^2+1}{2z}\right) + a^2} \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{e^{2i\theta}}{1 - 2\cos\theta + a^2} = \int \frac{z^2}{a^2 z - az^2 - a + z} \frac{dz}{iz}$$

$$= \frac{1}{i} \int \frac{z^2}{(az-1)(a-z)} dz$$

$$\int_0^{2\pi} \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} = \frac{1}{i} \int \frac{z^2}{(az-1)(a-z)} dz$$

Since  $C$  is the unit circle  $|z|=1$ .

$\therefore z = \frac{1}{a}$  &  $z = a$  are simple poles.

But  $-1 < a < 1$

$\therefore z = a$  lies within  $C$  &  $z = \frac{1}{a}$  lies outside it

$\therefore$  Residue of  $f(z)$  at  $z=a = \lim_{z \rightarrow a} (z-a) \frac{z^2}{i(az-1)(a-z)}$

$$= \lim_{z \rightarrow a} \frac{-z^2}{i(az-1)}$$

$$= \frac{-a^2}{i(a^2-1)}$$

$\therefore$  By Cauchy Residue theorem

$$\int_0^{2\pi i} \frac{e^{2i\theta}}{1+2a\cos\theta+a^2} d\theta = 2\pi i \left( \frac{-a^2}{a^2-1} \right)$$

$$= 2\pi i \left( \frac{-a^2}{a^2-1} \right)$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta + i \sin 2\theta}{1-2a\cos\theta+a^2} d\theta = \frac{-2\pi a^2}{a^2-1}$$

Equating real parts we get

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta = \frac{2\pi a^2}{1-a^2} //$$

(d) Let  $x = -0.4y + 6.4$  is a regression  $x$  on  $y$   $\therefore b_{xy} = -0.4$   
 Let  $y = -0.6x + 4.6$  is a regression equation  $y$  on  $x$ .  
 $\therefore b_{yx} = -0.6$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{-0.4 \times -0.6} = 0.4899$$

$\therefore$  Correlation Co-efficient = 0.4899.

e)  $f(x) = kx^2 e^{-x} \quad x > 0$

$$E(x) = \int_0^{\infty} x \cdot f(x) dx$$

$$= \int_0^{\infty} x \cdot kx^2 e^{-x} dx$$

$$= k \int_0^{\infty} x^3 e^{-x} dx$$

$$= k \left[ \frac{x^3 e^{-x}}{-1} - 3x^2 e^{-x} + 6x \frac{e^{-x}}{-1} - 6 e^{-x} \right]_0^{\infty}$$

$$= k [ 0 - (0 - 0 - 0 - 6) ]$$

$$= 6k //$$

To Find  $k$ .

We know if  $X$  is P.d.F then

$$\int_0^{\infty} f(x) dx = 1$$

$$\int_0^{\infty} kx^2 e^{-x} dx = 1$$

$$k \left[ \frac{x^2 e^{-x}}{-1} - 2x e^{-x} + \frac{2 e^{-x}}{-1} \right]_0^{\infty} = 1$$

$$k [ 0 - (0 - 0 - 2) ] = 1$$

$$2k = 1 \Rightarrow k = 1/2$$

$$\therefore E(x) = \frac{6k}{3} = 6(1/2)$$