# Solution of (EM IV) EXTC.

Supplementary Examination.

Q1)

a) Let u & v be two vectors then cauchy schwartz inequality is given by $|u \cdot v| \leq \|u\| \cdot \|v\|$.

Let $u = (a, b, c)$ & $v = (1, 1, 1)$

$\therefore \quad u \cdot v = a + b + c.$

$|u \cdot v| = a + b + c.$

$\|u\| = \sqrt{a^2 + b^2 + c^2} \qquad \|v\| = \sqrt{3}$

$\therefore$ By cauchy schwartz inequality

$|u \cdot v| \leq \|u\| \cdot \|v\|$

$a + b + c \leq \sqrt{a^2 + b^2 + c^2}\ \sqrt{3}$

$\therefore (a + b + c)^2 \leq 3\ (a^2 + b^2 + c^2)$

(b) Given:- $T_0 = 0$ is transmitted $\quad T_1 = 1$ is transmitted

$P(T_0) = 0.4$

$\therefore\ P(T_1) = 1 - P(T_0) = 1 - 0.4 = 0.6$

~~Rat~~ Let $R_0$ = 0 is Received.

$R_1$ = 1 is Received.

$\therefore\ P(R_0 / T_0)$ = 0 is recieved when 0 is transmitted $= 0.95$

$P(R_1 / T_0)$ = 1 is recieved when 0 is transmitted $= 1 - 0.95$
$= 0.05$

$P(R_1 / T_1)$ = 1 is received when 1 is transmitted $= 0.90$.

$P(R_0 / T_1)$ = 0 is Recieved when 1 is transmitted $= 1 - 0.90$
$= 0.1$

i) P( 1 is received) = $P(R_1/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$.

$P(R_1) = 0.9 \times 0.6 + 0.05 \times 0.4$

$P(R_1) = 0.56$.

ii) P( 1 was transmitted / 1 as Received). = ?

$$P(T_1/R_1) = \frac{P(R_1/T_1)\cdot P(T_1)}{P(R_1)}$$

$$= \frac{0.9 \times 0.6}{0.56} = 0.9642.$$

c) $\int_C \frac{\log z}{(z-1)^3} dz$ where C is $|z-1| = 1/2$.

Solution:- $|z-1| = 1/2$ is a circle with centre (1,0) & radius 1/2

$(z-1)^3 = 0$.

$z = 1, 1, 1$

$\therefore$ z = 1, lies inside the given curve C

By Cauchy Integral theorem.

$$\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0).$$

n=3

$$\therefore \int \frac{\log z}{(z-1)^3} = \frac{2\pi i}{(3-1)!} f^2(1).$$

$f(z) = \log z$.

$f'(z) = \frac{1}{z}$

$f''(z) = \frac{-1}{z^2}$.

$$\therefore \frac{2\pi i}{2} \left(\frac{-1}{z^2}\right)_{z=1}$$

$$\int \frac{\log z}{(z-1)^3} = -\pi i$$

(d) $F = xy + y^2 - 2y^2y'$.

F contain both $x, y, \& y'$.

$\therefore$ By Euler's equation we have.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$[x + 2y - 4yy'] - \frac{d}{dx}(-2y^2) = 0.$$

$$x + 2y - 4yy' + 4yy' = 0$$

$$x + 2y = 0.$$

$$y = -x/2.$$

Q2)

a) $\lambda$:

$P = 0.05$

$q = 1 - P = 1 - 0.05 = 0.95$ $\quad m = np = 20 \times 0.05 = 1$

$n = 20$.

$\therefore X = $ No. of Defective.

P( Packets containing exactly 2) and P(at least 2 defective) = ?

$\Rightarrow$

i) P(Packets containing exactly 2) $= P(X=2)$

$= {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$= {}^{n}C_{2} \cdot (0.05)^{2} (0.95)^{n-2}$

$= {}^{20}C_{2}\, (0.05)^{2}\, (0.95)^{18}$

$= 0.19$

ii) P(at least 2 defective) $= 1 - P(X \leq 1)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [{}^{n}C_{0}\, p^{0} q^{n-0} + {}^{n}C_{1}\, p^{1} q^{n-1}]$

$= 1 - [{}^{20}C_{0} (0.05)^{0} \cdot (0.95)^{20} + {}^{20}C_{1} (0.05)^{1} (0.95)^{19}]$

$= 1 - [0.36 + 0.38]$

$= 0.26$.

$\therefore$ No. of Packets containing exactly 2 defective $= NP$

$= 1000 \times 0.19$

$= 190$

No. of packets containing at least 2 defective $= NP$

$= 1000 \times 0.26$

$= 260$.

By poisson distribution

$P(X=2) = \dfrac{e^{-1}\, 1^{2}}{2!} = 0.1839.$

No. of Exactly 2 defective = 183

$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$

$= 1 - \left(\dfrac{e^{-1} 1^{0}}{0!} + \dfrac{e^{-1} 1^{1}}{1!}\right) =$

Q2b) $f(z) = \dfrac{2z-3}{z^2-4z-3}$

$$= \frac{2z-3}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$\therefore \Rightarrow \quad 2z-3 = A(z-3) + B(z-1)$

Put $z=1$      Put $z=3$

$A = 1/2$      $B = 3/2$

$$\therefore f(z) = \frac{1/2}{z-1} + \frac{3/2}{z-3}$$

$$= \frac{1}{2}\left[\frac{1}{z-1} + \frac{3}{z-3}\right] = \frac{1}{2}\left[\frac{1}{[(z-4)+3]} + \frac{3}{[(z-4)+1]}\right]$$

case(i) when $|z-4| < 1$.

$$f(z) = \frac{1}{2}\left[\frac{1}{3\left[1+\frac{z-4}{3}\right]} + \frac{3}{[1+(z-4)]}\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\left[1+\left(\frac{z-4}{3}\right)\right]^{-1} + 3\left[1+(z-4)\right]^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\left(1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2 + \cdots\right] + 3\left[1-(z-4)+(z-4)^2-(z-4)^3 \cdots\right]$$

$$= \frac{1}{6}\left[1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2+\cdots\right] + \frac{3}{2}\left[1-(z-4)+(z-4)^2+\cdots\right]$$

Required Taylor series.

case ii) $|z-4| > 3$

$$f(z) = \frac{1}{2}\left[\frac{1}{[(z-4)+3]} + \frac{3}{[(z-4)+1]}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(z-4)\left[1+\frac{3}{z-4}\right]} + \frac{3}{(z+4)\left[1+\frac{1}{z-4}\right]}\right]$$

$= \frac{1}{2}\left[ \frac{1}{z-4}\left[1+\frac{3}{z-4}\right]^{-1} + \frac{3}{z-4}\left[1+\frac{1}{z-4}\right]^{-1}\right]$

$= \frac{1}{2}\left[ \frac{1}{z-4}\left(1 - \frac{3}{z-4} + \left(\frac{3}{z-4}\right)^2 - \left(\frac{3}{z-4}\right)^3 + \cdots\right]\right.$

$\left. + \frac{3}{z-4}\left[1 - \frac{1}{z-4} + \left(\frac{1}{z-4}\right)^2 - \left(\frac{1}{z-4}\right)^3\right]\right]$

This is Required Laurent series.

(c) $\int_0^1 (y'^2 - 4y^2 + 2x^2y)\,dx$.

Let $y = c_0 + c_1x + c_2x^2$ $\quad y(0) = 0 \Rightarrow c_0 = 0$.

$y' = c_1 + 2c_2x$. $\quad y(1) = 0 \Rightarrow c_0 + c_1 + c_2 = 0$.

$c_1 = -c_2$

$\therefore y' = c_1 - 2c_1x$. $\quad y = c_1(x - x^2)$

$= c_1(1-2x)$.

$\therefore I = \int_0^1 \left[c_1^2(1-2x)^2 + 4(x-x^2)^2c_1^2 + 2x^2c_1(x-x^2)\right]dx$

$= \int_0^1 c_1^2(1-4x+4x^2) + 4(x^2-2x^3+x^4)c_1^2 + 2c_1(x^3-x^4)\,dx$.

$= c_1\left[c_1\left(x - \frac{4x^2}{2} + \frac{4x^3}{3}\right) + 4c_1\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right) + 2\left[\frac{x^4}{4} - \frac{x^5}{5}\right]\right]_0^1$

$= c_1\left[c_1\left(1 - \frac{4}{2} + \frac{4}{3}\right) + 4c_1\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + 2\left(\frac{1}{4} - \frac{1}{5}\right) - 0\right]$

$= c_1\left[\frac{c_1}{3} + \left(\frac{-8c_1}{15}\right) + \frac{1}{10}\right]$

$I \Rightarrow -\frac{c_1^2}{5} + \frac{1}{10}c_1$

$\frac{dI}{dc_1} = 0 \Rightarrow -\frac{2c_1}{5} + \frac{1}{10} = 0 \Rightarrow +\frac{2c_1}{5} = \frac{+1}{10}$

$c_1 = \frac{5}{20} = \frac{1}{4}$

$\therefore y = \frac{1}{4}[1-2x]$ //

Q3)

(a)

| Maths $X$ | Stats $Y$ | $x = X-\bar{X}$, $x = X-65$ | $y = Y-\bar{Y}$, $y = Y-66$ | $xy$ | $x^2$ | $y^2$ |
|---|---|---|---|---|---|---|
| 78 | 84 | 13 | 18 | 234 | 169 | 324 |
| 36 | 51 | -29 | -15 | 435 | 841 | 225 |
| 98 | 91 | 33 | 25 | 825 | 1089 | 625 |
| 25 | 60 | -40 | -6 | 240 | 1600 | 36 |
| 75 | 68 | 10 | 2 | 20 | 100 | 04 |
| 82 | 62 | 17 | -4 | -68 | 289 | 16 |
| 90 | 86 | 25 | 20 | 500 | 625 | 400 |
| 62 | 58 | -3 | -8 | 24 | 9 | 64 |
| 65 | 53 | 0 | -13 | 0 | 0 | 169 |
| 39 | 47 | -26 | -19 | 494 | 676 | 361 |
| | | 0 | 0 | 2704 | 5398 | 2224 |

$N = 10$

$\bar{x} = \frac{\Sigma X}{10} = 65$ $\quad \bar{y} = \frac{660}{10} = 66$

$$\therefore r = \frac{N\Sigma xy - \Sigma x \Sigma y}{\sqrt{N\Sigma x^2 - (\Sigma x)^2} \cdot \sqrt{N\Sigma y^2 - (\Sigma y)^2}} = \frac{10(2704) - 0}{\sqrt{10(5398)} \cdot \sqrt{10(2224)}}$$

$$= 0.7804.$$

∴ For Spearman's Rank correlation.

| X | Y | $R_1$ | $R_2$ | $d_i^2 = (R_1 - R_2)^2$ |
|---|---|---|---|---|
| 78 | 84 | 4 | 3 | 1. |
| 36 | 57 | 9 | 9 | 0 |
| 98 | 91 | 1 | 1 | 0 |
| 25 | 60 | 10 | 6 | 16 |
| 75 | 68 | 5 | 4 | 1 |
| 82 | 62 | 3 | 5 | 4 |
| 90 | 86 | 2 | 2 | 0 |
| 62 | 58 | 7 | 7 | 0 |
| 65 | 53 | 6 | 8 | 4 |
| 39 | 47 | 8 | 10 | 4 |
| | | | | 30. |

∴ $\sum d_i^2 = 31.$

$$R = 1 - \frac{6\sum d_i^2}{N^3 - N}$$

$$= 1 - \frac{6(30)}{10^3 - 10} = 0.8181$$

(b) $u_1 = (1,2,1) \quad u_2 = (1,1,3) \quad u_3 = (2,1,1)$

Let $u_1 = v_1$.

$\therefore\ v_1 = (1,2,1) \qquad \|v_1\| = \sqrt{6}$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} \cdot v_1$$

$$= (1,1,3) - \frac{\langle (1,1,3)\cdot(1,2,1)\rangle}{6}(1,2,1)$$

$$= (1,1,3) - \frac{6}{6}(1,2,1)$$

$$= (1,1,3) - (1,2,1)$$

$v_2 = (0,-1,2)$

$\|v_2\| = \sqrt{5}$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} \cdot v_2$$

$$= (2,1,1) - \frac{\langle (2,1,1),(1,2,1)\rangle}{6} v_1 - \frac{\langle (2,1,1)\cdot(0,-1,2)\rangle}{5} \cdot v_2$$

$$= (2,1,1) - \frac{5}{6}(1,2,1) - \frac{1}{5}(0,-1,2)$$

$$= (2,1,1) - \left(\frac{5}{6}, \frac{10}{6}, \frac{5}{6}\right) - \left(0, -\frac{1}{5}, \frac{2}{5}\right)$$

$$= \left(\frac{7}{6}, -\frac{7}{15}, \frac{7}{30}\right).$$

$$\|v_3\| = \sqrt{\frac{49}{30}} = \frac{7}{\sqrt{30}}$$

$\therefore$ orthonormal vectors are.

$$e_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$e_2 = \frac{v_2}{\|v_2\|} = \left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$e_3 = \frac{v_3}{\|v_3\|} = \left(\frac{\sqrt{30}}{6}, -\frac{\sqrt{30}}{15}, \frac{\sqrt{30}}{30}\right) = \left(\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$$

(3)(c) By divergence Theorem

$$\iint_S \vec{N} \cdot \vec{F}\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

Here $F = x^3 i + x^2 y j + x^2 z k$

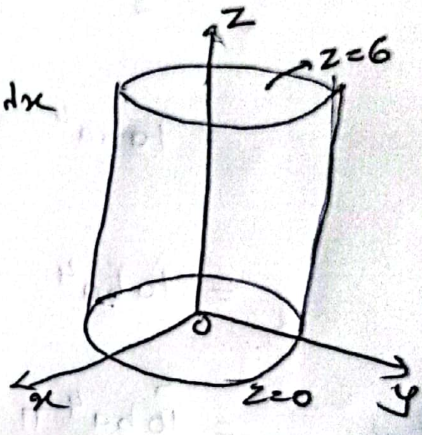
$$\nabla \cdot F = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)$$

$$= 3x^2 + x^2 + x^2$$

$$= 5x^2$$

$$\therefore \iiint_V \nabla \cdot F\, dV = \iiint_V 5x^2\, dx\, dy\, dz$$

To cover whole cylinder bounded $x^2+y^2=a^2$, $z=0$ & $z=b$ Varies from 0 to $b$, $y$ varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$ and $x$ varies from $-a$ to $a$.




$$\therefore \iiint 5x^2\, dx\, dy\, dz = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}} \int_0^b 5x^2\, dz\, dy\, dx$$

$$= 5 \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}} x^2 (b)\, dy\, dx$$

$$= 5b \int_{-a}^{a} x^2 (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}}\, dx$$

$$= 5b \int_{-a}^{a} x^2\, 2\sqrt{a^2-x^2}\, dx.$$

$$= 10b \int_{-a}^{a} x^2 \sqrt{a^2-x^2}\, dx$$

Put $x = at$ $\quad dx = a\,dt$

when $x = -a \Rightarrow t = -1$

$x = a \Rightarrow t = 1$

$$\iiint_V 5x^2 dV = 10b \int_{-1}^{1} a^2 t^2 \sqrt{a^2 - a^2 t^2}\, a\, dt$$

$$= 20b \int_0^1 a^4 t^2 \sqrt{1-t^2}\, dt$$

$$= 20ba^4 \int_0^1 t^2 \sqrt{1-t^2}\, dt$$

Put $t^2 = u$

$2t\,dt = du \Rightarrow dt = \dfrac{du}{2\sqrt{u}}$

Limit remain same.

$$\iiint_V 5x^2 dV = \int^{1} 20ba^4 \int_0^1 u\,(1-u)^{1/2} \frac{du}{2\sqrt{u}}$$

$$= \frac{20ba^4}{2} \int_0^1 u^{1/2} (1-u)^{1/2} du = 10ba^4 \beta(3/2, 3/2)$$

$\because \int_0^1 x^m (1-x)^n$
$\beta(m+1, n+1)$

$$= 10ba^4 \frac{\Gamma 3/2\, \Gamma 3/2}{\Gamma 3}$$

$$= 10ba^4 \frac{\frac{1}{2}\cdot\frac{1}{2}\pi}{2}$$

$$= \frac{\cancel{10}^{5} ba^4 \pi}{\cancel{8}4}$$

$$\iint_S N\cdot\bar{F}\, dS = \frac{5a^4 b\pi}{4}$$

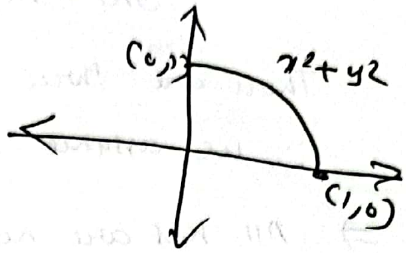
Q4)

(a) $\int F \cdot dr = ?$

$F = \cos y\, i - x \sin y\, j$    $C$ is the curve $y = \sqrt{1-x^2}$.

$x^2 + y^2 = 1$

$\therefore \int F \cdot dr = \int_C (\cos y\, i - x \sin j)(dx i + j dy)$




$= \int_C \cos y\, dx - x \sin y\, dy$

$= \left[ x \cos y - 0 \right]_{(1,0)}^{(0,1)}$

$= (0 - 1) = -1.$

(b) i) $\{1, x, 1+x+x^2\}$

Sol<sup></sup>n: Let $v_1 = 1$, $v_2 = x$, $v_3 = 1+x+x^2$.

Consider.

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$.

$k_1 + k_2 x + k_3(1+x+x^2) = 0 + 0 \cdot x + 0 \cdot x^2$

$k_1 + k_2 x + k_3 + k_3 x + k_3 x^2 = 0 + 0 \cdot x + 0 \cdot x^2$

$(k_1 + k_3) + (k_2 + k_3) x + k_3 x^2 = 0 + 0 \cdot x + 0 \cdot x^2$

$\Rightarrow k_3 = 0$

$\Rightarrow k_2 + k_3 = 0$

$\Rightarrow k_2 = 0$

$\Rightarrow k_1 + k_3 = 0$

$\Rightarrow k_1 = 0$

All $k$'s are zero.

Hence $\{1, x, 1+x+x^2\}$ is Linearly Independent

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ii) $v_1 = (2,3)$ + $v_2 = (-4,3)$ $v_3 = (1,5)$

consider $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$k_1(2,3) + k_2(-4,3) + k_3(1,5) = (0,0)$$

$$2k_1 - 4k_2 + k_3 = 0.$$
$$3k_1 + 3k_2 + 5k_3 = 0.$$

There are three variable + two equation

$\therefore$ we assume $k_3 = t$, $t \neq 0$.

$\Rightarrow$ All $k_i$'s are not zero.

Hence $\{u_1, u_2, u_3\}$ are linearly dependent.

(c) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2} d\theta$, $-1<a<1$.

$\underline{\underline{\text{soln}}}$ let $z = e^{i\theta}$ $\therefore dz = ie^{i\theta} d\theta$

$$\cos\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$= \frac{z - \bar{z}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}.$$

consider $\displaystyle\int_0^{2\pi} \frac{e^{2i\theta}}{1-2a\cos\theta + a^2} d\theta = \int \frac{z^2}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{e^{2i\theta}}{1-2\cos\theta+a^2} = \int \frac{z^{\cancel{3}}}{a^2z - az^2 - a + z} \frac{dz}{i\cancel{z}}$$

$$= \frac{1}{i}\int \frac{z^2}{(az-1)(a-z)} dz$$

$$\int_0^{2\pi} \frac{e^{2i\theta}}{1-2a\cos\theta+a^2} = \frac{1}{i}\int \frac{z^2}{(az-1)(a-z)} dz$$

Since $C$ is the unit circle $|z| = 1$.

$\therefore$ $z = \frac{1}{a}$ & $z = a$ are simple poles.

But $-1 < a < 1$

$\therefore$ $z = a$ lies within $c$ & $z = \frac{1}{a}$ lies outside it

$\therefore$ Residue of $f(z)$ at $z = a$ = $\lim_{z \to a} (z-a) \frac{z^2}{i(az-1)(a-z)}$

$$= \lim_{z \to a} \frac{-z^2}{i(az-1)}$$

$$= \frac{-a^2}{i(a^2-1)}$$

$\therefore$ By Cauchy Residue theorem

$$\int_0^{2\pi i} \frac{e^{2i\theta}}{1+2a\cos\theta + a^2} d\theta = 2\pi \cancel{i} \frac{1}{\cancel{i}}\left(\frac{-a^2}{a^2-1}\right)$$

$$= 2\pi \left(\frac{-a^2}{a^2-1}\right)$$

$$\therefore \int_0^{2\pi i} \frac{\cos 2\theta + i\sin 2\theta}{1-2a\cos\theta + a^2} d\theta = \frac{-2\pi a^2}{a^2-1}$$

Equating Real parts we get

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2} = \frac{2\pi a^2}{1-a^2} \text{ //.}$$

(d) Let $x = -0.4y + 6.4$ is a regression $x$ on $y$ $\therefore b_{xy} = -0.4$

Let $y = -0.6x + 4.6$ is a regression equation $y$ on $x$.

$\therefore b_{yx} = -0.6$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{-0.4 \times -0.6} = 0.4899.$$

$\therefore$ correlation Co-efficient $= -0.4899$.

e) $f(x) = kx^2 e^{-x}$ $\quad x \geq 0$.

$$E(X) = \int_0^\infty x \cdot f(x)\, dx.$$

$$= \int_0^\infty x \cdot kx^2 e^{-x}\, dx$$

$$= k\int_0^\infty x^3 e^{-x}\, dx$$

$$= k\left[ x^3 \frac{e^{-x}}{-1} - 3x^2 e^{-x} + 6x \frac{e^{-x}}{-1} \div 6 e^{-x} \right]_0^\infty$$

$$= k\left[0 - (0 - 0 \div 0 \div 6)\right]$$

$$= 6k$$

To Find $k$.

we know if $X$ is p.d.f then

$$\int_0^\infty f(x)\, dx = 1$$

$$\int_0^\infty k x^2 e^{-x}\, dx = 1$$

$$k\left[ x^2 \frac{e^{-x}}{-1} - 2x e^{-x} + \frac{2e^{-x}}{-1} \right]_0^\infty = 1$$

$$k\left[0 - (0 - 0 - 2)\right] = 1$$

$2k = 1 \Rightarrow k = 1/2$ $\quad \therefore E(X) = 6k = 6(1/2) = 3$